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ON SOME TYPES OF SINGULAR INTEGRAL EQUATIONS OCCURRING IN APPLICATIONS

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Integral equations whose kernels are complicated by the presence besides the well known component, the so-called Cauchy kernel, of an additional term with nonintegrable singularities at the ends of the variation interval of the independent variable, are investigated in some detail. Integrals of this kind occur in investigations of a group of special mixed problems in the potential and elasticity theories. Possible approaches to the interpretation of particular equations, not only with symmetric limits (differing only by their signs) of integration, are outlined. Such equations undoubtedly deserve attention owing to the interest in them in the applied field.

1. Let us analyze in detail the solutions of the following equation of the first kind:

$$\frac{1}{\pi i}\int_{\gamma_0}\omega(t)\left[\frac{1}{t-t_0}-\lambda\frac{1}{t-\rho^2/t_0}\right]dt = f(t_0), \quad -\rho\leqslant t_0\leqslant\rho \tag{1.1}$$

where λ is a parameter which may assume any real or complex values, and the free term $f(t_0)$ can be any arbitrarily specified Hölder function on the closed real segment $\gamma_0\ (-\rho\leqslant t_0\leqslant\rho)$.

The distinctive method developed here for analyzing a group of mixed problems of the potential theory has considerably assisted (in its advanced development stage) in the discovery of means for transforming singular integral equations of a particular structure and, then, in revealing properties of their solutions.

Note that in the present investigation the harmonic transformation of the completed process is almost entirely omitted and only verification and refinement of the results obtained by it are presented. The starting point of the investigation which allowed subsequent analysis of Eq. (1.1) was the observation on a mixed problem of the theory of potential for a semicircle.

Equation (1.1) was first considered for real values of parameter λ within the limits $-1<\lambda<1$ (the limit cases of $\lambda=\pm1$ require separate analysis). Setting $\lambda=\cos\theta$ we were entitled to assume $0<\theta/\pi<1$; it appeared expedient to introduce in the addition to parameter θ the constant $\alpha=1-\theta/\pi$ (the solution is extended to the case of any real and complex λ. (see Sect. 3)).

Equation (1.1) has generally solutions of two kinds. One of these, as shown subsequently, is continuous in the closed interval $-\rho\leqslant t\leqslant\rho$ and vanishes at its ends

$t = \pm \rho)$. Its structure is defined thus:

$$\omega(t_0) = \frac{1}{2\pi i}\int_{\gamma_0} \chi(\alpha; t, t_0) f(t)\left[\frac{1}{t-t_0} - \frac{1}{t-\rho^2/t_0}\right] dt \tag{1.2}$$

$$\chi(t, t_0) = \left(\frac{\rho - t_0}{-\rho - t_0}\right)^{\alpha}\left(\frac{\rho - t}{-\rho - t}\right)^{-\alpha} + \left(\frac{\rho - t_0}{-\rho - t_0}\right)^{-\alpha}\left(\frac{\rho - t}{-\rho - t}\right)^{\alpha}$$

It is, however, valid only when the supplementary condition

$$i \operatorname{ctg}\frac{\pi\alpha}{2}\Lambda[\alpha; f(t)] = 0 \tag{1.3}$$

$$\Lambda[\alpha; f(t)] = \frac{1}{2\pi i}\int_{\gamma_0}\left[e^{-\pi i\alpha}\left(\frac{\rho - t}{-\rho - t}\right)^{\alpha} - e^{\pi i\alpha}\left(\frac{\rho - t}{-\rho - t}\right)^{-\alpha}\right]\frac{f(t)}{t}\, dt$$

is satisfied by the additional term $f(t)$ (the limit cases $\alpha = 0$ and $\alpha = 1$ are not considered here).

This solution appeared in [1] without proper substantiation. Unfortunately, the multiplier $i \operatorname{ctg}(\pi\alpha/2)$ at the integral in the limiting condition (1.3) was omitted. However it is important only in the limit cases of $\alpha = 0$ and $\alpha = 1$ which (as well as in the related cases of $\theta = \pi$ and $\theta = 0$) require special consideration.

The second qualitatively different discontinuous solution (with integrable singularities at points $t = \pm\rho$ was obtained much later and is adduced here for the first time. It is of the form

$$\mu(t_0) = \frac{1}{2\pi i}\int_{\gamma_0}\chi\left(\frac{\theta}{\pi}; t, t_0\right) f(t)\left(\frac{1}{t-t_0} + \frac{1}{t-\rho^2/t_0}\right) dt \mp \tag{1.4}$$

$$e^{\pm i\theta}\left(\frac{\rho - t_0}{-\rho - t_0}\right)^{\mp\theta/\pi}\Lambda\left(\frac{\theta}{\pi}; f(t)\right) + 2i\sin\vartheta\, A(\vartheta)\times$$

$$\left[e^{-i\vartheta}\left(\frac{\rho - t_0}{-\rho - t_0}\right)^{\vartheta/\pi} + e^{i\theta}\left(\frac{\rho - t_0}{-\rho - t_0}\right)^{-\vartheta/\pi}\right], \quad -\rho < t_0 < \rho$$

where the binomial with the constant multiplier $A(\upsilon)$ is the solution of the homogeneous equation (1.1) (when $f(t) = 0$). It should obviously not be lost out of sight but without necessarily maintaining it in formula (1.4). (That solution of the homogeneous equation (1.1) was also presented in [1] in a different form. The supposedly another different form of it appeared there by misunderstanding).

We adduce some of the integral formulas that may be required in testing final conclusions

$$\frac{1}{2\pi i}\int_{\gamma_0}\left(\frac{\rho - t}{-\rho - t}\right)^{\pm\alpha}\frac{dt}{t - z} = \mp\frac{ie^{\pm\pi i\alpha}}{2\sin\pi\alpha}\left[\left(\frac{\rho - z}{-\rho - z}\right)^{\pm\alpha} - 1\right] \tag{1.5}$$

$$\frac{1}{2\pi i}\int_{\gamma_0}\left(\frac{\rho - t}{-\rho - t}\right)^{\pm\alpha}\frac{dt}{t - t_0} = \mp\frac{i}{2}\left[\operatorname{ctg}\pi\alpha\left(\frac{\rho - t_0}{-\rho - t_0}\right)^{\pm\alpha} - \frac{e^{\pm\pi i\alpha}}{\sin\pi\alpha}\right]$$

$$\frac{1}{2\pi i}\int_{\gamma_0}\left(\frac{\rho - t}{-\rho - t}\right)^{\pm\alpha}\frac{dt}{t - \rho^2/t_0} = \mp\frac{i}{2\sin\pi\alpha}\left[\left(\frac{\rho - t_0}{-\rho - t_0}\right)^{\pm\alpha} - e^{\pm\pi i\alpha}\right]$$

$$\frac{1}{2\pi i}\int_{\gamma_0}\left(\frac{\rho - t}{-\rho - t}\right)^{\pm\alpha}\frac{dt}{t} = \pm\frac{i}{2}e^{\pm\pi i\alpha}\operatorname{tg}\frac{\pi\alpha}{2}$$

where z is the affix of any arbitrary point lying outside the segment γ_0, and point t_0 is located on γ_0 so that the second (in sequence) integral on the left is taken in the sense of the principal value.

The linear-fractional power function appearing under the integral sign is the limit value of function

$$\left(\frac{\rho-z}{-\rho-z}\right)^{\pm\alpha} = \exp\left(\pm\alpha\ln\frac{\rho-z}{-\rho-z}\right) \to \left(\frac{\rho-t_0}{-\rho-t_0}\right)^{\pm\alpha} \quad (z\to t_0)$$

with z tending to point t_0 of the upper side of the cut drawn along segment γ_0; simultaneously

$$\ln\frac{\rho-z}{-\rho-z} = \int\limits_{\gamma_0}\frac{dt}{t-z}, \quad \left(\ln\frac{\rho-z}{-\rho-z}\right)_{z\to 0} = \pi i$$

We also introduce the function

$$\ln\frac{\rho-\rho^2/z}{-\rho-\rho^2/z} = \int\limits_{\gamma_0}\frac{dt}{t-\rho^2/z}$$

regular outside the rays $\gamma_1\,(-\rho, -\infty;\ \infty, \rho)$ (which complement γ_0 to the whole of the real axis).

In what follows a significant part is played by the relationships

$$\left(\frac{\rho-\rho^2/z}{-\rho-\rho^2/z}\right)^{\pm\alpha} = \left(\frac{\rho-z}{-\rho-z}\right)^{\pm\alpha} \times \begin{cases} e^{\mp\pi i\alpha}, & \operatorname{Im} z > 0 \\ e^{\pm\pi i\alpha}, & \operatorname{Im} z < 0 \end{cases} \tag{1.6}$$

It will be seen that the relation between these functions is different in different half-planes of variation of variable z.

R e m a r k 1. If the specified free term of Eq. (1.1) automatically reduces functional (1.3) to zero, then formulas (1.2) and (1.4) yield two solutions of Eq. (1.1). One of these is continuous in the closed interval $-\rho \leqslant t_0 \leqslant \rho$ and vanishes at its ends, and the second has at the ends integral singularities. The two solutions differ from one another by some solution of the homogeneous equation (1.1) (within the constant factor).

R e m a r k 2. It will be readily seen that for the free term of Eq. (1.1) equal to some constant C the related to it in conformity with (1.2) density $\omega(t)$ becomes identically zero; hence the constant C must also (in conformity with (1.3)) be made equal zero. This becomes at once clear if we take into account, besides formula (1.2), the integral relationships (1.5). As previously indicated, formula (1.2) yields only those solutions of Eq. (1.1) that are continuous on the closed segment γ_0, which is only possible when its specified right-hand side satisfies the limiting condition (1.3). This makes it at once clear that, when the problem is that of deriving (1.2) as the solution for density, the free term of Eq. (1.1), which is a constant quantity, can only be zero, and, consequently, the density itself must unavoidably become identically zero. However, the solution of the other type (defined by formula (1.4)) is necessarily nonzero when the free term of Eq. (1.1) is constant. Let us assume it to be unity, then in this simplest case the solution, as can be readily seen, is provided by any of the following formulas:

$$\mu(t_0) = \frac{i}{2\sin\theta}\left[e^{i\theta}\left(\frac{\rho-t_0}{-\rho-t_0}\right)^{-\theta/\pi} - e^{-i\theta}\left(\frac{\rho-t_0}{-\rho-t_0}\right)^{\theta/\pi}\right] \mp$$

$$i \operatorname{tg} \frac{\theta}{2} e^{\pm i\theta} \left(\frac{\rho - t_0}{-\rho - t_0}\right)^{\mp\theta/\pi}$$

which differ from each other by some solution of the same homogeneous equation.

2. It can be shown that the density $\omega(t)$ determined by formula (1.2) with any Hölder function $f(t)$ actually satisfies Eq. (1.1) only with an accuracy to some constant defined by the functional appearing on the left of the first of equalities (1.3). (Any function $f(t)$ supplemented by the functional on the left-hand side of the first of equalities (1.3) reduces the function in the second of these equalities to zero). This density $\mu(t)$, unlike the other, is not only bounded on the closed segment γ_0 but, also, vanishes at its ends $t_0 = \pm\rho$. The author had checked this by obtaining solution (1.2) for comparatively simple particular values of the free term. Validity of this statement can be easily ascertained also in the general case. To simplify the proof we assume the specified function $f(t)$ to be differentiable the required number of times.

We represent the sought density (1.2) in the form

$$\begin{aligned} &\omega(t_0) = f(t_0)\, p(\alpha; t_0) + q(\alpha; t_0) \\ &p(\alpha; t_0) = \frac{1}{2\pi i} \int_{\gamma_0} \chi(\alpha; t, t_0) \left[\frac{1}{t - t_0} - \frac{1}{t - \rho^2/t_0}\right] dt \\ &q(\alpha; t_0) = \left(t_0 - \frac{\rho^2}{t_0}\right) \frac{1}{2\pi i} \int_{\gamma_0} \chi(\alpha; t, t_0)\, g(t, t_0) \frac{dt}{t - \rho^2/t_0} \\ &g(t, t_0) = \frac{f(t) - f(t_0)}{t - t_0} \end{aligned} \tag{2.1}$$

The expression for function $q(\alpha; t_0)$ can be written thus:

$$q(\alpha, t_0) = \left(t_0 - \frac{\rho^2}{t_0}\right) \left\{\frac{1}{2\pi i} \int_{\gamma_0} \chi(\alpha; t, t_0) \frac{g(t, t_0) - g(t_0, t_0)}{t - \rho^2/t_0} dt + g(t_0, t_0)\, r(\alpha; t_0)\right\}$$

where the newly introduced function

$$r(\alpha; t_0) = \frac{1}{2\pi i} \int_{\gamma_0} \chi(\alpha; t, t_0) \frac{dt}{t - \rho^2/t_0}$$

can be represented, after calculating its integral, as

$$r(\alpha; t_0) = -\frac{i}{2 \sin \pi\alpha} \left[e^{-\pi i\alpha} \left(\frac{\rho - t_0}{-\rho - t_0}\right)^{\alpha} - e^{\pi i\alpha} \left(\frac{\rho - t_0}{-\rho - t_0}\right)^{-\alpha}\right]$$

Surveying the elementary (in comparison with other used subsequently) formulas (1.5) and (1.6) and taking into account the definition of function $\chi(\alpha; t, t_0)$, we conclude that $p(\alpha; t_0) = 0$. As the result we obtain the simpler (in comparison with the initial) formula

$$\omega(t_0) = q(\alpha; t_0) = \left(t_0 - \frac{\rho^2}{t_0}\right) \left\{\frac{1}{2\pi i} \int_{\gamma_0} \chi(\alpha; t, t_0)\, h(t, t_0) \frac{t - t_0}{t - \rho^2/t_0} dt + g(t_0, t_0)\, r(\alpha; t_0)\right\} \tag{2.2}$$

$$h(t, t_0) = \frac{g(t, t_0) - g(t_0, t_0)}{t - t_0}$$

where function $h(t_1, t_0)$ is continuous on γ_0 owing to the condition imposed on $f(t)$.

To determine the behavior of the (linear fractional with respect to t) function which appears here as the third factor in the integrand and (nonlinearly) dependent also on the variable t_0 it is sufficient to observe its variation near the ends of segment γ_0 ; in other words for the following simultaneous values of their affixes: $t = \pm(\rho - \varepsilon)$ and $t_0 = \pm(\rho - \varepsilon_0)$, where ε and ε_0 are as small as desired positive quantities. The behavior of the considered function in the vicinity of these affixes is determined (with higher order smalls taken into account) by the equality

$$\left|\frac{t - t_0}{t - \rho^2 / t_0}\right| = |\varepsilon - \varepsilon_0| \left[\varepsilon + \varepsilon_0 + \rho \sum_{\nu=2}^{\infty} \left(\frac{\varepsilon_0}{\rho}\right)^{\nu}\right]^{-1} \tag{2.3}$$

which shows it does not exceed in modulus unity.

Hence the considered function remains bounded on the closed interval γ_0 and, consequently, the integral in (2.2) converges absolutely. This implies that the density $\omega(t)$ vanishes at the ends of segment γ_0 $(0 < \alpha < 1)$.

Thus the assumption of reality of parameter λ strictly varying within the limits $-1 < \lambda < 1$ has predetermined (together with other factors) the possibility of realizing the way for deriving the solution of Eq. (1.1), outlined by the author.

3. Let us now assume that parameter λ may have any real or complex values so that $\lambda = \lambda_1 + i\lambda_2$, where λ_1 and λ_2 are real (positive or negative) numbers. It is expedient to retain the previous representation of parameter λ, although this time as the cosine of the complex argument $\vartheta = \vartheta_1 + i\vartheta_2$ with real ϑ_1 and ϑ_2 defined in terms of the specified λ_1 and λ_2. It is advisable to introduce besides the complex argument ϑ the complex quantity $\alpha = 1 - \vartheta/\pi$. The basic formula

$$\lambda = \cos\vartheta = \cos\vartheta_1 \operatorname{ch}\vartheta_2 - i\sin\vartheta_1 \operatorname{sh}\vartheta_2 \tag{3.1}$$

splits into two real equalities

$$\lambda_1 = \cos\vartheta_1 \operatorname{ch}\vartheta_2, \quad \lambda_2 = -\sin\vartheta_1 \operatorname{sh}\vartheta_2 \tag{3.2}$$

At first glance it may seem that independently of these or other parameters λ_1 and λ_2 (arbitrarily specified) it is not possible to maintain the same, or more precisely, nonnegative the sign of $\sin\vartheta_1$, on the contrary it seems probable that the sign of this quantity (when passing from one λ to some other) changes. Meanwhile, the attainment of the aim, as will become clear in the following, of the intended extension appears possible when the nonnegative sign of $\sin\vartheta_1$ is determined beforehand, in other words for the real component of argument ϑ_1 within the limits $0 < \vartheta_1 < \pi$. (The possibility of such selection of the variation interval of ϑ_1 is also important becauce then the real part of parameter $\alpha = 1 - \vartheta/\pi$ is less than unity, with the exception of the limit case $\vartheta_1 = 0$ which requires special consideration). Looking somewhat ahead, we would point out that it is, nevertheless, possible to fix the interval $0 < \vartheta_1 < \pi$ (in conformity with ones objectives) which absorbs any value of ϑ_1 (defined by the formulas adduced below) that correspond to any desired (real) parameters λ_1 and λ_2.

Eliminating from (3.2) the quantities ϑ_2, we obtain the equation

$$\xi^2 + (\lambda_1^2 + \lambda_2^2 - 1)\,\xi - \lambda_2^2 = 0, \quad \xi = \sin^2\vartheta_1 \tag{3.3}$$

In considering this equation we have to deal with the radical

$$R = [(\lambda_1^2 + \lambda_2^2 - 1)^2 + 4\lambda_2^2]^{1/2} = [(\lambda_1^2 + \lambda_2^2 + 1)^2 - 4\lambda_1^2]^{1/2} \tag{3.4}$$

and bear in mind the following evident inequalities:

$$|\lambda_1^2 + \lambda_2^2 - 1\,| \leqslant R \leqslant \lambda_1^2 + \lambda_2^2 + 1 \tag{3.5}$$

Here and throughout the following analysis the radical containing nonegative quantities are taken at their arithmetic values.

For any υ_1 contained within the indicated limits, from (3.3) we obtain

$$\sin\theta_1 = [-{}^1/_2\,(\lambda_1^2 + \lambda_2^2 - 1) + {}^1/_2 R]^{1/2} \tag{3.6}$$

in which for obvious reasons we take R with the plus sign. This formula is consistent; the expression in square brackets does not exceed unity, which can be readily seen from the first of inequalities (3.5).

It would be possible to first derive, instead (3.3), the equation

$$\eta^2 - (\lambda_1^2 + \lambda_2^2 + 1)\,\eta + \lambda_1^2 = 0, \quad \eta = \cos^2\upsilon_1$$

which yields

$$\cos\upsilon_1 = \pm\,[{}^1/_2\,(\lambda_1^2 + \lambda_2^2 + 1) - {}^1/_2 R]^{1/2} \tag{3.7}$$

At this stage R is taken with the minus sign, since otherwise the expression in square brackets according to the left inequality in (3.5) (when $\lambda_1^2 + \lambda_2^2 \geqslant 1$ as well as $\lambda_1^2 + \lambda_2^2 < 1$) would certainly exceed unity (being equal unity in the first case only when $\lambda_1 = \pm 1$ and $\lambda_2 = 0$ and $R = 0$).

The expression in square brackets in (3.7) is obviously nonnegative and, conveniently, does not exceed unity. This is evident from the expression for R in (3.4) and from the left inequality in (3.5).

Passing now to the first of equalities (3.2) and taking into account that the quantity $\operatorname{ch}\vartheta_2$ is positive for any (real) ϑ_2, we conclude that it is necessary in formula (3.7) to leave at the external radical the same sign as the intrinsic sign of the real component λ_1. This is very important. In fact, taking as the base the sign of the right-hand side of (3.7), as proposed above, it is already possible (using the nonnegative $\sin\vartheta_1$ yielded by (3.6)) to fix uniquely the argument υ_1 by directly taking it from the interval $0 \leqslant \upsilon_1 < \pi$.

It should be stressed that fixing the magnitude and sign of the pure imaginary component of argument ϑ is entirely unambiguous, since by the second of equalities (3.2) it is obvious that a sign opposite to that of the pure imaginary component of the specified complex parameter λ is to be assigned to component ϑ_2. Let us now set

$$\delta = \frac{\lambda_1}{\cos\vartheta_1} = \operatorname{ch}\vartheta_2, \quad \eta = -\frac{\lambda_2}{\sin\theta_1} = \operatorname{sh}\vartheta_2 \tag{3.8}$$

$$h = \frac{\delta}{\eta}, \quad \sqrt{\delta^2 - \eta^2} = 1$$

where δ and η are to be assumed known.

It is further desirable to ascertain that the (positive) quantity δ is in fact (in accordance with the first of equalities (3.8)) smaller than unity. This is perceived from the readily verified inequality

$$\left(\frac{\lambda_1}{\cos \vartheta_1}\right)^2 - 1 = \frac{1}{2}[(\lambda_1^2 + \lambda_2^2 - 1) + R] \geqslant 0$$

Since $\delta \geqslant 1$, the first equality implies that

$$\vartheta_2 = \ln (\delta \pm \sqrt{\delta^2 - 1}) \tag{3.9}$$

where the plus sign is taken for $\vartheta_2 > 0$ and minus for $\vartheta_2 \leqslant 0$, with ϑ_2 vanishing for $\delta = 1$.

After conventional transformations we obtain

$$h^2 = 1 + \frac{2}{(\lambda_1^2 + \lambda_2^2 - 1) + R}$$

which implies that $|h| > 1$ $(\delta > |\eta|)$.

It is admissible to determine component ϑ_2 in a form somewhat different from (3.9) by composing the ratio of the first of equalities (3.8) to the second. We have

$$\vartheta_2 = \frac{1}{2} \ln \frac{h+1}{h-1} \tag{3.10}$$

which shows that ϑ_2 is positive when $h > 1$, and negative when $h < -1$. Formulas (3.9) and (3.10) are evidently adequate.

Thus the considered here more complex variant differs from the original $[-1 < \lambda < 1]$, and what is important, from the considerably widened variation range of parameters ϑ (α).

4. With a more or less thorough understanding of the developed process as a whole the thought about the validity of interpreting the same (1.) and (1.4) formulas for any (including also complex) constant λ becomes clearer and more persistent. And here the true adjustment of the tentatively formulated final conclusion is obtained, as previously [2], simply by direct formal substitution in the singular equation (1.1) of the values of densities yielded by relations (1.2) and (1.4).

Let us actually substitute alternately into the input equation (1.1) the two different expressions (1.2) and (1.4) for the unknown density. In both cases we obtain fairly complicated singular multiple integrals whose content is difficult to grasp and each of which splits in turn into a number of simply complex multiple integrals. We had encountered them in a somewhat veiled form already when investigating the initial case of $-1 < \lambda < 1$, i.e. for real ϑ and α (the author had obtained singular multiple integrals of an appropriately altered structure in [2]). These were subjected to a radical transformation consisting of changing the sequence of ordinary integrals in each of them. Then the inner ordinary integrals contained in the thus newly formed double singular integrals were determined in closed form. As the result, the input multiple integrals were reduced to the more elementary simple integrals with singularities of the same type. (As already pointed out, these relationships were in fact used previously, although not explicitly shown in systematic formulation).

It will be readily seen that the expressions for the above multiple integrals now derived in this manner are externally of the same form as the previous ones, except for the (implicitly contained in them) complexity of parameters ϑ and α .

We again stress that these formulas, now derived without deviation from, and in exact correspondence with formulas (1.2), the first of equalities (3.2), and formula (3.7), in no way reduce the freedom of choice of α and ϑ (unavoidably supplemented by conditions $0 < \mathrm{Re}^{\vartheta} / \pi < 1$ and $0 < \mathrm{Re}\,\alpha < 1$ of which the second is invariably present only when the first is satisfied). This is fairly evident, since Eq. (1.1) does not contain self-conjugate values of density, but only the unknown density itself.

Formulas for the transformation of specific double integrals to single integrals of the same type (which had not been presented anywhere before and are given here in full, since they are required for carrying out proper calculations) are of the form

$$\frac{1}{2\pi i}\int_{\gamma_0}\left(\frac{\rho-t}{-\rho-t}\right)^{\pm\alpha}\frac{dt}{t-t_0}\,\frac{1}{2\pi i}\int_{\gamma_0} g(t_1)\left(\frac{\rho-t_1}{-\rho-t_1}\right)^{\mp\alpha}\frac{dt_1}{t_1-t}=$$
$$\frac{1}{4}g(t_0)\pm$$
$$\frac{\operatorname{ctg}\pi\alpha}{4\pi}\int_{\gamma_0} g(t)\left[1-\left(\frac{\rho-t_0}{-\rho-t_0}\right)^{\pm\alpha}\left(\frac{\rho-t}{-\rho-t}\right)^{\mp\alpha}\right]\frac{dt}{t-t_0}$$

$$\frac{1}{2\pi i}\int_{\gamma_0}\left(\frac{\rho-t}{-\rho-t}\right)^{\pm\alpha}\frac{dt}{t-\rho^2/t_0}\,\frac{1}{2\pi i}\int_{\gamma_0} g(t_1)\left(\frac{\rho-t_1}{-\rho-t_1}\right)^{\mp\alpha}\frac{dt_1}{t_1-t}=$$
$$\pm\frac{1}{4\pi}\int_{\gamma_0} g(t)\left[\operatorname{ctg}\pi\alpha-\right.$$
$$\left.\frac{1}{\sin\pi\alpha}\left(\frac{\rho-t_0}{-\rho-t_0}\right)^{\pm\alpha}\left(\frac{\rho-t}{-\rho-t}\right)^{\mp\alpha}\right]\frac{dt}{t-\rho^2/t_0}$$

$$\frac{1}{2\pi i}\int_{\gamma_0}\left(\frac{\rho-t}{-\rho-t}\right)^{\pm\alpha}\frac{dt}{t-\rho^2/t_0}\,\frac{1}{2\pi i}\int_{\gamma_0} g(t_1)\left(\frac{\rho-t_1}{-\rho-t_1}\right)^{\mp\alpha}\frac{dt_1}{t_1-\rho^2/t}=$$
$$\mp\frac{1}{4\pi\sin\pi\alpha}\int_{\gamma_0} g(t)\left\{\left[\left(\frac{\rho-t_0}{-\rho-t_0}\right)^{\pm\alpha}\left(\frac{\rho-t}{-\rho-t}\right)^{\mp\alpha}-1\right]\frac{1}{t-t_0}-\right.$$
$$\left.\left[e^{\pm\pi i\alpha}\left(\frac{\rho-t}{-\rho-t}\right)^{\mp\alpha}-1\right]\frac{1}{t}\right\}dt$$

$$\frac{1}{2\pi i}\int_{\gamma_0}\left(\frac{\rho-t}{-\rho-t}\right)^{\pm\alpha}\frac{dt}{t-t_0}\,\frac{1}{2\pi i}\int_{\gamma_0} g(t_1)\left(\frac{\rho-t_1}{-\rho-t_1}\right)^{\mp\alpha}\frac{dt_1}{t_1-\rho^2/t}=$$

$$\mp\frac{1}{4\pi}\int_{\gamma_0} g(t)\left\{\left[\operatorname{ctg}\pi\alpha\left(\frac{\rho-t_0}{-\rho-t_0}\right)^{\pm\alpha}\left(\frac{\rho-t}{-\rho-t}\right)^{\mp\alpha}-\frac{1}{\sin\pi\alpha}\right]\times\right.$$
$$\left.\frac{1}{t-\rho^2/t_0}-\frac{1}{\sin\pi\alpha}\left[1-e^{\pm\pi i\alpha}\left(\frac{\rho-t}{-\rho-t}\right)^{\mp\alpha}\right]\frac{1}{t}\right\}dt$$

A survey of these formulas immediately shows that they actually do not loose their meaning and remain unaltered also for complex ϑ (α) under the same, not for the first time stated, conditions that limit the variation range of their real parts. The indicated integral relations are useful for checking the validity of the resolving formulas (1.2) and (1.4), when in addition to (4.1) one must bear in mind the formulas derived from these by the formal substitution of θ / π for α .

This is the position when $\operatorname{Re}(\vartheta/\pi)$ and $\operatorname{Re}(\alpha)$ are numerically less than unity. In the mutually exclusive limit cases, when $\operatorname{Re}(\vartheta/\pi) = 1$ or $\operatorname{Re}(\alpha) = 1$ as the resolving formula corresponding to each of them we take the one that is adapted to the other parameter, subject to the limiting condition $\operatorname{Re}(\alpha) = 0$ or $\operatorname{Re}(\vartheta/\pi) = 0$, respectively. (It is impled that formula (1.2) is the solution of Eq. (1.1) also when $\operatorname{Re}(\alpha) = 0$, which occurs when (1.3) is satisfied, and it is only then that the above assertion is entirely valid). Individual investigation of the cases of $\operatorname{Re}(\vartheta/\pi) = 1$ and $\operatorname{Re}(\alpha) = 1$ is somewhat difficult.

5. Let us indicate a possible approach to the analysis of singular integral equations of a certain type; more complex equations of this kind appear, as a rule, in the analysis of some special mixed problems of the two-dimensional theory of elasticity in special configuration regions.

To the uninitiated reader the basic idea of the proposed method may appear somewhat artificial and exaggerated. However the process of its application in practice is simple and its basic feactures are understandable and profitable, although the establishment and formalization of the idea itself was slow and difficult. This could have been due to that the problem was approached from distinctly altered and new positions, as compared with those the author had adhered in his earlier work related to the development of similar methods for application to fundamental problems of the theory of elasticity.

Let us assume that the right-hand half-plane of the variable $z = x + iy$ is slit along the real segment γ of length a issuing from the coordinate origin. For the semi-infinite region formed thus on the right, which we shall denote by S, we formulate the following boundary value problem. It is required to determine the pair of functions $\varphi_1(z)$ and $\psi_1(z)$ which are regular in that region and vanishingly small in absolute value in its remote part, using the following boundary conditions. At the upper and lower edges of the γ-slit the following conditions are, respectively, specified:

$$\varkappa\varphi_1^+(t) + \overline{\psi_1^+(t)} = f(t), \quad \varkappa\varphi_1^-(t) + \overline{\psi_1^-(t)} = f(t); \quad 0 < t < a \tag{5.1}$$

where $\varkappa$ is some, generally, complex constant, and $f(t)$ is the Hölder function specified on that segment. These functions are subjected along the imaginary axis to the boundary condition of the fairly conventional type

$$\varphi_1(t) + \overline{\psi_1(t)} = 0, \quad t = iy \quad (-\infty < y < \infty) \tag{5.2}$$

The elementary function

$$w = \sqrt{a^2 - z^2} \tag{5.3}$$

yields (with its suitably selected branch, $w^+ = a$ when $z = 0$) the conformal map of region S onto the lower half-plane of variable w. We then have at the upper and lower edges of the slit $w^\pm(x) = \pm\sqrt{a^2 - x^2}$, $0 < x < a$, and on the imaginary axis $w(\pm iy) = \pm\sqrt{a^2 + y^2}$, $0 < y < \infty$.

Using this conformally mapping function, the thus formulated problem can be reduced to the comparatively easily analyzed Karleman equation (since the input boundary conditions do not contain derivatives of the unknown functions). Bearing in mind a similar possibility, and in conformity with our aims, we shall continue the analysis of the same boundary value problem in a different setting.

Adding and subtracting the limiting equalities (5. 1) term-by-term one from the other, we obtain relationships of the kind

$$\varkappa\,[\varphi_1^+(t)+\varphi_1^-(t)]+\overline{[\psi_1^+(t)+\psi_1^-(t)]}=2f(t) \tag{5.4}$$

$$\varkappa\,[\varphi_1^+(t)-\varphi_1^-(t)]+\overline{[\varphi_1^+(t)-\psi_1^-(t)]}=0, \quad 0<t<a \tag{5.5}$$

or equivalent to them. We add to these the equality which determines the auxilliary function $\omega(t)$ and is introduced on the same segment γ by formula

$$\varkappa\,[\varphi_1^+(t)-\varphi_1^-(t)]-\overline{[\psi_1^+(t)-\psi_1^-(t)]}=2\omega(t), \qquad 0<t<a \tag{5.6}$$

The purpose of this ancilliary function by far exceeds the limits of explanation within the scope of the particular problem of the potential theory. The reasons for its introduction in the analysis in the form (5. 6) will become clear subsequently.

Adding the boundary condition (5. 5) (in the already altered form) term-by-term to the just introduced ancilliary relation (5. 6) and then subtracting these equalities term-by-term one from the other, we obtain

$$\varphi_1^+(t)-\varphi_1^-(t)=\frac{1}{\varkappa}\,\omega(t) \ \text{ on } \ \gamma, \quad 0<t<a \tag{5.7}$$

$$\overline{\psi_1^+(t)}-\overline{\psi_1^-(t)}=-\,\omega(t) \ \text{ on } \ \gamma, \quad 0<t<a$$

It is expedient to represent equality (5. 7) in the form

$$\varphi_1^+(t_0)-\frac{1}{2\pi i\varkappa}\int\limits_\gamma\frac{\omega(t)}{t-z}\,dt=\varphi^-_1(t_0)-\frac{1}{2\pi i\varkappa}\int\limits_\gamma\frac{\omega(t)}{t-z}\,dt \ \text{ on } \ \gamma$$

$$z\to t_0 \text{ above } \gamma \qquad\qquad z\to t_0 \text{ below } \gamma$$

We introduce the new function $\varphi(z)$ which is regular in region S in conformity with the equality

$$\varphi(z)=\varphi_1(z)-\frac{1}{2\pi i\varkappa}\int\limits_\gamma\frac{\omega(t)}{t-z}\,dt \tag{5.8}$$

from which follows that

$$\varphi^+(t_0)=\varphi^-(t_0) \ \text{ on } \ \gamma$$

Hence function $\varphi(z)$ can be continued through segment γ, and is regular everywhere in the right-hand half-plane; it will be seen that in the remote parts of the region the absolute value of this function can be made as small as desired.

Dealing in a similar manner with the second basic equality (5. 7), we introduce in the same right-hand half-plane the function

$$\psi(z)=\psi_1(z)+\frac{1}{2\pi i}\int\limits_\gamma\frac{\overline{\omega(t)}}{t-z}\,dt \tag{5.9}$$

which also vanishes at infinity.

We now substitute $\varphi(z)$ and $\psi(z)$ for functions $\varphi_1(z)$ and $\psi_1(z)$ determined by (5.8) and (5.9), respecively, in the boundary condition (5.2) (satisfied along the imaginary axis) which, after this transformation assumes the form

$$\varphi(t_0)+\overline{\psi(t_0)}=-\frac{1}{2\pi i\varkappa}\int_\gamma\frac{\omega(t)}{t-t_0}dt-\frac{1}{2\pi i}\int_\gamma\frac{\omega(t)}{t+t_0}dt$$
$$t_0=iy\quad(-\infty<y<\infty)$$

From this we obtain for these functions representations of the form

$$\varphi(z)=-\frac{1}{2\pi i}\int_\gamma\frac{\omega(t)}{t+z}dt,\quad \psi(z)=\frac{1}{2\pi i\bar{\varkappa}}\int_\gamma\frac{\overline{\omega(t)}}{t+z}dt$$

Taking these equalities into consideration, from (5.8) and (5.9) we obtain for the unknown functions the following expressions:

$$\varphi_1(z)=\frac{1}{2\pi i\varkappa}\int_\gamma\frac{\omega(t)}{t-z}dt-\frac{1}{2\pi i}\int_\gamma\frac{\omega(t)}{t+z}dt \tag{5.10}$$
$$\psi_1(z)=-\frac{1}{2\pi i}\int_\gamma\frac{\overline{\omega(t)}}{t-z}dt+\frac{1}{2\pi i\bar{\varkappa}}\int_\gamma\frac{\overline{\omega(t)}}{t+z}dt$$

Passing in the first of these to the limit $z\to t_0$ above and below segment γ and adding term-by-term the obtained equalities (after multiplication by $\varkappa$), we obtain

$$\varkappa[\varphi^+_1(t_0)+\varphi_1^-(t_0)]=\frac{1}{\pi i}\int_\gamma\frac{\omega(t)}{t-t_0}dt-\frac{\varkappa}{\pi i}\int_\gamma\frac{\omega(t)}{t+t_0}dt\quad\text{on }\gamma$$

Carrying out similar elementary transformations on the limit values of function $\psi_1(z)$ defined by formula (5.10), we obtain an equality of a structure similar to the previous one, whose conjugate equality is of the form

$$\overline{\psi_1^+(t_0)}+\overline{\psi^-(t_0)}=\frac{1}{\pi i}\int_\gamma\frac{\omega(t_0)}{t-t_0}dt-\frac{1}{\pi i\varkappa}\int_\gamma\frac{\omega(t)}{t+t_0}dt\quad\text{on }\gamma$$

Applying the so far unused limiting condition (in the modifed form) (5.4)we obtain the singular integral equation which satisfies the ancillary function $\omega(t)$, and convince ourselves that this equation is of the form

$$\frac{1}{\pi i}\int_\gamma\omega(t)\left(\frac{1}{t-t_0}-\lambda\frac{1}{t+t_0}\right)dt=f(t_0),\quad\lambda=\frac{\varkappa^2+1}{2\varkappa} \tag{5.11}$$

(the required value of parameter λ is obtained by a suitable of parameter $\varkappa$).

Let us now assume that Eq. (5.11) has been somehow resolved and the obtained from it ancilliary function $\omega(t)$ is introduced in the respective integrals in formulas (5.10). This would yield the two functions $\varphi_1(z)$ and $\psi_1(z)$ which solve the boundary value problem.

As already noted, the same problem can be solved comparatively simply using the conformal mapping function; in this context the following is also true. Having determined by using, as indicated, the mapping function (5.3) the solution of the boundary value problem by the straightforward conditions (5.1) and (5.2), we compose the left-hand side of equality (5.6) using the obtained functions $\varphi_1(z)$ and $\psi_1(z)$. The derived formula determines the doubled value of density $\omega(t)$ which exactly satisfies the singular equation (5.11).

It should be noted that for the three particular values of function $f(t)$ for which the integral equation (5.11) yields to effective analysis, it is possible to complete quickly the study of the considered boundary value problem using directly formulas (5.11). We recall that the same equation (5.11) was considered in [3] in a different way in relation to real values of parameter λ.

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